SOLUTION OF BOUNDARY VALUE PROBLEM USING SHOOTING METHOD

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ABSTRACT: - Most of the problems in the world use mathematical tools for its solutions. Some problems have analytical solutions. However analytical solutions of all the problems do not exist. Almost all the problems in the world can be solved using numerical methods. Boundary Value Problem (BVP[1]) is defined as a mathematical problem in which the partial differential equation is satisfied with in a certain, and in addition certain conditions are to be satisfied on the boundary of the region. Shooting Method [2] is an iterative method used for solving BVP. In this method, BVP is first converted into Initial Value Problem by assuming the required number of conditions at the initial points. Shooting method is more time efficient than other numerical methods available like Range-Kutta[3] Method, Newton Raphson[4] Method, Secant Method[5] and fastly converges toward the solution.

KEYWORDS: Numerical Methods, Boundary Value Problem,, Shooting Method, Initial Value Problem

1. INTRODUCTION

Many of the problems in physical sciences are governed by differential equation. When the partial differential equation is satisfied with in a certain, and in addition certain conditions are satisfied on the boundary of the region, the problem is said to be a Boundary Value Problem. A non-trivial solution y(x) of the equation:

$$y'' + \lambda y = 0$$

that satisfies the boundary conditions y(a)=0 and y(b)=0, have an entire solution. For this, one has to satisfy one condition at each of two distinct values of x.

1.1 Initial Value Problem: Consider a problem

$$y'' + y = 0$$

with boundary conditions y(1) = 3 and y'(1) = -4. Such types of problems are called **initial value problems**[6]. Both of these boundary conditions relate to one value of x, namely x=1.

1.2 Some Initial Value Problems:

a) Vibrations of an Infinite String:

Consider a wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}, \qquad -\infty < x < \infty, t > 0$$

with initial conditions

$$U(x,0) = F(x), \; U_t(x,0) = G(x)$$

where y = F(x) is the initial position of the string and G(x) is the initial velocity at the point x.

b) Dirichlet Problem[7] (for the upper half plane) : Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad -\infty < x < \infty, y > 0$$

$$u(x,0) = f(x)$$

with the conditions that u is bounded as $y \to \infty$, u and u_x vanish as $|x| \to \infty$.

1.3 Some Boundary Value Problems:

a) $u'' = u + x^2$

with boundary conditions

$u(0) = 1$ and $u(1) = 0$

b) $y'' + k y = x^2, \qquad 0 < x < 1$

with boundary conditions

$y(0) = 0, \; y'(1) = 1$

2. SHOOTING METHOD

Shooting method can be used in this paper for solving boundary value problems. As initial value problems are easy to solve, so by converting a BVP into an initial value problem by assuming the required number of conditions at the initial point. Then the conditions at the other end are satisfied by varying the assumed conditions at the initial point. Before taking up the shooting method for the boundary value problem, the initial value problem is briefly discussed as follow.

In the initial value problem, first we discuss about what is initial value problem. A general solution an ordinary differential equation

$\phi(t,y,y',y''.............y^m) = 0$ **.....(2.1.1)**

is a relation between y, t and m arbitrarily constants which satisfies the equation, but which contains no derivatives . The solution may be an implicit relation of the form

$$w(t,y,C_1,C_2,..............C_m) = 0$$

or an explicit function of t of the form

$$y=w\ (t,C_1,C_2,.............C_m)$$

The m arbitrary constants C_1, C_2, -----------, C_m can be determined by prescribing m conditions of the form

$y^s(t_0)=n_s$, **where** $s=0,1,2,..................m-1$ **......(2.1.2)**

at one point $t=t_0$ are called initial condition.

The differential equation (2,1,1) together with the initial condition (2.1.2) is called m^{th} order initial value problem. Now the m^{th} order differential equation

$y^m =F(t,y,y'.............y^{m-1})$ **..........(2.1.3)**

with initial conditions may be written as equivalent system of m first order initial value problem.

$u_1=y$
$u_1=u_2$
$u_2=u_3$
- - - -
- - - -
- - - -
$u''_{m-1}=u_m$
$u''_n=F(t,u_1,u_2,............u_m)$
$u_1(t_0)=\eta_0, u_2(t_0)=\eta_1............., u_m(t_0)=\eta_{m-1}$

which in vector notation becomes

$u'=f\ (t,u)$
$u(t_0)=\eta$
......(2.1.4)
where
$u=[u_1,u_2,- - - - - u_m]^T$
$f=[u_2,u_3,- - - - u_m f]^T$
$n=[\eta_0, \eta_1,...... \eta_{m-n}]$

Thus the method solution of the first order initial value problem.

$\frac{du}{dt}=f(t,u)$ where $u(t_0)=\eta$

may be used to solve the system of first order value problem (2.1.4) & the m^{th} order initial 2.2 **Solution of Shooting Method:**

We solve initial value problems $u''=f(x,u,u')$

$u(a)=r_1,\ u'(a)=\alpha$

Where α is same approximation of the initial slope. Using any of the method for solving the initial value problems, the approximation $u^{(1)}(b)$ to the solution $u(b)$ is determined. This value is either smaller or larger than the required solution $u(b)= r_2$.

Let us denote $g(\alpha_0)= u^{(1)}(b) – u(b)$

where α_0 is the first approximation of α, if $g(\alpha_0)= 0$ then the condition is satisfied at x = b. If condition is not satisfied, then we repeat the above procedure using $u'(a) = \alpha_1$ to find another estimate $u^{(2)}(b)$ for $u(b)$. The process is usually repeated initial the computed value at x = b agree with the boundary condition u(b). Depending on the choice of u'(a), the computed solution may, overshoot or undershoot, the required solution as shown in the following figure.

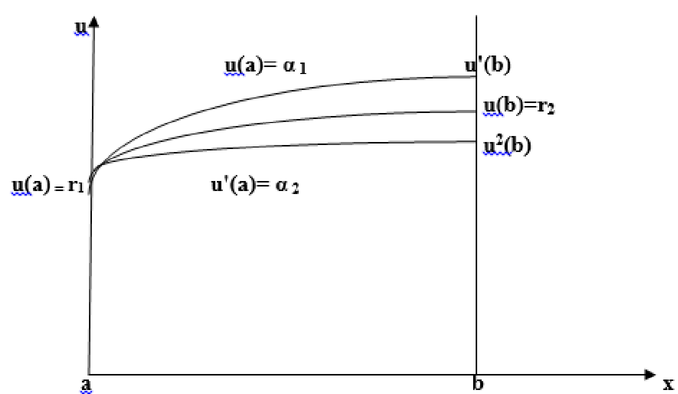

Fig.2.1: Solution by Shooting Method

Shooting method may be described as produce which defines a functional relationship $g(\alpha)=0$, between u(b) and the initial slope u' (a) as given in the following figure:

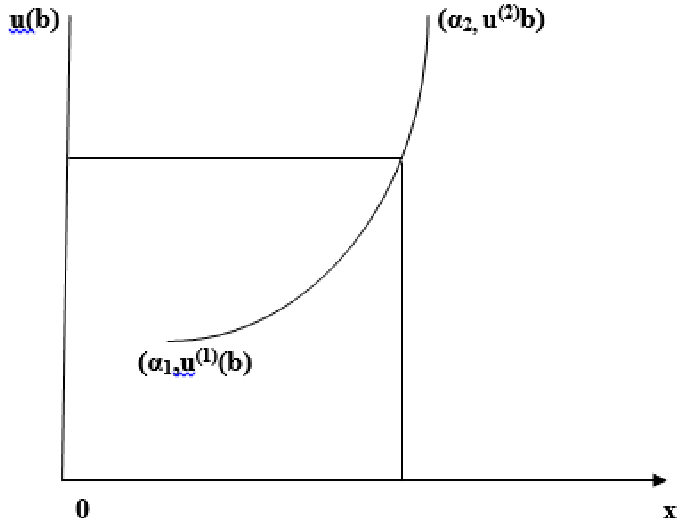

Fig.2.2: Functional relationship between the final value and the initial slope.

The problem is then to find the root of this equation. The root cannot be determined by the Newton-Rabhson Method and after using the secant method, one gets

$$\alpha_{n+1} = \alpha_n - \frac{[\alpha_n - \alpha_{n-1}]}{[g(\alpha_n) - g(\alpha_{n-1})]} \times g(\alpha_n), \quad n=1, 2, 3, \ldots\ldots\ldots\ldots\ldots..$$

If the differential equation is linear, then the Shooting Method becomes very simple. It can be shown that the function relationship $g(\alpha)=0$ between $u'(a)$ and $u(b)$ is also linear. Suppose that one have computed to solution $u_1(x)$ and $u_2(x)$ of the differential equation. Both the solutions are obtained using the same initial value $u_1(a) = r_1$, $u_2(a) = r_2$ but different initial slopes $u_1(a)$ and $u_2(a)$. Then, by the superposition principle the solution of the differential equation can be written as

$u(x)=c_1u_1(x)+c_2u_2(x)$ .... (2. 2. 1)

we have

$u(a)=r_1=c_1r_1+c_2r_2$ .... (2. 2. 2)

or

$c_1+c_2=1$

and

$u(b)= r_2 =c_1u_1(b)+c_2u_2(b)$ .... (2. 2. 3)

solving (2 2. 2) and (2. 2. 3) we get

$c_2=r_2-u_1(b)/u_2(b)-u_1(b)$, $c_1=1-c_2$ .... (2. 2. 4)

substituting (2. 2. 4) in (2. 2. 1) one get the solution of the differential equation.

3. CONCLUSION

This research paper shows how fastly and accurately the shooting method solves boundary value problem using ordinary differential equation of second or higher order. This method works well for the problems in which variation in the dependent variable is not large, even in the case, where variations are not large, the method of satisfying the boundary condition at the other end is difficult and often results in indetermination to keep the values within the bounds or use.

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