Associative Binary Operations on a Set with Four Elements

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Abstract— **Associativity of a binary operation is one of most fundamental property in algebra. In this paper, the authors discuss binary operations on a four-element set and show, by partition and composition of mapping, that exactly 3492 operations out of the 4294967296 existing binary operations on the set are associative with counting that how many composition tables are to be verified.**

Keywords— **binary operations; associativity; isomorphism; order; Partition;**
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I. INTRODUCTION

The number of binary operation on a set of only four elements is as large as 4,294,957,296. To prove this is an easy calculation; there are four different answers for each of the 16 seats of a 4X4 operation table so the number of distinct binary operations is 4^{16} . The number of commutative operations on the 4-set can calculated in a similar way, resulting in 4^{10}=1,048,576 binary operations. This calculation is straightforward, but no easy calculation and no educated guess, seems to give the answer to following question:-

"How many binary operations on a four elements are associative?"

The objective of this paper is to answer this question. In other words, how many of the 4,294,957,296 different binary operations on four-element set are associative.

II. GENERAL CONCEPTS

A *binary operation* (hereafter referred to only as an operation) on a set S is a rule that assigns to each ordered pair (a, b), where a and b are elements of S, exactly one element, denoted by ab, in S. A operation on a set is *associative* if x (yz) = (xy) z for every x, y and z in S.

An *isomorphism* between S and S' is a one-to-one function Φ mapping S onto S' such that Φ(xy)= Φ(x)Φ(y) for all x and y in S. If there exists an isomorphism between S and S', then S and S' are said to be *isomorphic*, denoted by S≈S'.

Order of an element is cardinality of set generated by that element.

III. USEFUL THEOREM

The following theorem about sets, S and S', closed under operations are well known and easy to prove:-
Theorem: - If there exists an isomorphism between S and S' and the operation on S is associative then an operation on S' is also associative.

IV. OPERATIONS ON A SET WITH TWO ELEMENT

Number of binary operation on a set of two elements is 16. Number of associative binary operation on a set of two elements is 8.See [2].

V. OPERATIONS ON A SET WITH THREE ELEMENT

Number of binary operation on a set of two elements is 19683. Number of associative binary operation on a set of two elements is 113.See [2],[3].

VI. OPERATIONS ON A SET WITH FOUR-ELEMENT

As mentioned in the introduction, the number of possible binary operations on a set of four elements is 4,294,957,296. Of these 1,048,576 are commutative. We now proceed to answer the question: How many binary operations on a four elements are associative?

For a four-element set S proving associatively for a given operation amounts to verify different equations (xy) z=x (yz), where x, y, z are elements of S. A single counterexample suffices to show that a given operation is not associative. Clearly, counterexamples need not to be unique.

Now we discuss alternative ways to check the associativity. Define Φ_a from (S, *) into (S,*), by $\Phi_a(x)$ =a*x for all x in S, where a belong to S. If composition of Φ_a with Φ_b is equal to Φ_{a*b} for all a, b in S then we say that S is associatively under the binary operation *.
Explanation: - $a*(b*c) = \Phi_a(b*c) = \Phi_a(\Phi_b(c))$ …..(1)
$(a*b)*c = \Phi_{a*b}(c)$ …..(2)
From (1) and (2) we conclude that above definition is equivalent with definition of associatively of binary operation * on set S. See[3]

VII. ALGORITHM FOR FINDING NUMBER OF ASSOCIATIVE BINARY OPERATION ON N-ELEMENT

Algorithm given below is taken from Amit Sehgal & Manmohan [3] and Sarita & Amit Sehgal [4].The analysis of the associative binary operations on n-element set S will now divide into 3 steps:-

(i) Partition the set of n^n mappings such a way that element of same partition can be obtained by using one-one and onto mapping from S onto S.
(ii) Rearrange the partition according to their order of any element of the partition.(Say order as k).
(iii) Calculate the contribution towards number of associative binary operations when one row is fill

by the first element of i^{th} partition which can also fill k-1 more rows and remaining row n-k can be filled by i^{th} and onwards partitions (if any) with two conditions given below.

Conditions: - Before starting calculation, firstly we insured that no table associative table counted twice. For this, we make some rules:-
(i) If we fixed r^{th} row from any element i^{th} partition (which has order k) then we can fill at least k-1 more rows. Remaining unfilled rows can by filled with element of i^{th} and onwards partition (if any) however selected element of i^{th} partition cannot fill the unfilled rows before selected row.
(ii) If table contains n different entry of i^{th} partition then contribution towards number of associative operations counted is 1/n.

Explanation (How partition reduce the checking the associative work):- Firstly, we prove that Table-1 and **Table-2** given below are isomorphic.

*	a	b	c
a	b	a	a
b			
c			

Table-1

*	a	b	c
a			
b	b	c	b
c			

Table-2

Here, if we define an isomorphic mapping from S to S by f (a) =b, f (b) =c, f(c) =a. Then we get f(b)=f(a)f(a) , f(a)=f(a)f(b), f(a)=f(a)f(c).Then composition table after applying above isomorphic mapping to Table-1, we get table-2. Hence, table-1 and table-2 are isomorphic.

Similarly, we can prove that if baa comes in second row is isomorphic to bcb in third row and if baa comes in third row is isomorphic to bcb in first row. Hence, table with atleast one row baa is isomorphic to table with atleast one row bcb (row number of bcb may be different from row number of baa).

Hence, there is no need to check associative composition table for bcb when associative composition table baa is checked because they give isomorphic composition table. (By theorem stated in section 3 if a composition table with baa is associative then isomorphic composition having row bcb is also associative).

As above discussion, if we apply any one-one and onto to baa we get {baa, ccb, bab, bcb, cca, caa} which forms a partition and now our calculation is reduced six times(out of six elements now we have to check associativity only for single element).

On the basis of above algorithm, now we proceed towards finding number of associative operations on a set with three elements.

A. *Step 1st*

If set S has n elements, then total number of mapping possible from set S to S= n^n
In our problem n=4, then total number of mapping possible from set S to S= 256 .

Here we consider S={0,1,2,3} and first digit of each mapping comes from 0, second comes from 1, third comes from 2 and last comes from 3.

Partition No	First element of Partition	Total number of element in partition
1	0000	4
2	0001	24
3	0003	12
4	0011	12
5	0012	24
6	0013	24
7	0022	12
8	0023	12
9	0032	12
10	0123	1
11	0132	6
12	0211	24
13	0231	8
14	1000	12
15	1001	12
16	1002	24
17	1032	3
18	1200	24
19	1230	6

B. *Step 2nd*

Partition No	First element of Partition	Order of an element of partition	Generated elements belongs to partition
1	1230	4	1,14,1,19
2	0012	3	2,6,18
3	1002	3	3,16,12
4	1200	3	4,4,17
5	0231	3	5,5,19
6	0001	2	6,18
7	0013	2	7,15
8	0211	2	8,17
9	0011	2	9,18
10	0032	2	10,17
11	1000	2	11,15
12	1001	2	12,16
13	0132	2	13,19
14	1032	2	14,19
15	0003	1	15
16	0022	1	16
17	0023	1	17
18	0000	1	18
19	0123	1	19

C. *Step 3rd*

1).*When i=1(assume element from partition is 1230)*

Number of elements in Ist Partition are 6

Assumed row for 1230	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0.5×6=3
2nd	1	0.5×6=3
3rd	1	0.5×6=3
4th	1	0.5×6=3
Total	**4**	**12**

2).*When i=2(assume element from partition is 0012)*

Number of elements in 2nd Partition are 24

Assumed row for 0012	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×24=0
2nd	1	0×24=0
3rd	250	1×24=24
4th	1	1×24=24
Total	**253**	**48**

3).*When i=3(assume element from partition is 1002)*

Number of elements in 3rd Partition are 24

Assumed row for 1002	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×24=0
2nd	1	0×24=0
3rd	226	2×24=48
4th	1	1×24=24
Total	**229**	**72**

4).*When i=4(assume element from partition is 1200)*

Number of elements in 4th Partition are 24

Assumed row for 1200	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	202	1×24=24
2nd	202	0.5×24=12
3rd	202	0.5×24=12
4th	1	0×24=0
Total	**607**	**48**

5).*When i=5(assume element from partition is 0231)*

Number of elements in 5th Partition are 8

Assumed row for 0231	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×8=0
2nd	178	0.5×8=4
3rd	178	0.5×8=4
4th	178	0.5×8=4
Total	**535**	**12**

6).*When i=6(assume element from partition is 0001)*

Number of elements in 6th Partition are 24

Assumed row for 0001	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×24=0
2nd	1700	22×24=528
3rd	1360	4.5×24=108
4th	170	4.5×24=108
Total	**3231**	**744**

7).*When i=7(assume element from partition is 0013)*

Number of elements in 7th Partition are 24

Assumed row for 0013	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×24=0
2nd	438	8×24=192
3rd	146	3×24=72
4th	1	0×24=0
Total	**586**	**264**

8).*When i=8 (assume element from partition is 0211)*

Number of elements in 8th Partition are 24

Assumed row for 0211	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×24=0
2nd	244	2×24=48
3rd	244	1×24=24
4th	122	0×24=0
Total	**611**	**72**

9).*When i=9 (assume element from partition is 0011)*

Number of elements in 9th Partition are 12

Assumed row for 0011	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×12=0
2nd	490	18×12=216
3rd	98	2×12=24
4th	98	1×12=12
Total	**687**	**252**

10).*When i=10 (assume element from partition is 0032)*

Number of elements in 10th Partition are 12

Assumed row for 0032	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1	0×12=0
2nd	1	0×12=0
3rd	172	3×12=36
4th	172	3×12=36
Total	**346**	**72**

11).*When i=11 (assume element from partition is 1000)*

Number of elements in 11th Partition are 12

Assumed row for 1000	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	296	14×12=168
2nd	148	1×12=12
3rd	74	0×12=0
4th	74	0×12=0
Total	**592**	**180**

12).*When i=12 (assume element from partition is 1001)*

Number of elements in 12th Partition are 12

Assumed row for 1001	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	248	4.5×12=54
2nd	124	4.5×12=54
3rd	62	0×12=0
4th	62	0×12=0
Total	**496**	**108**

13).*When i=13 (assume element from partition is 0132)*

Number of elements in 13th Partition are 6

Assumed row for 0132	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	50	0×6=0
2nd	50	0×6=0
3rd	150	9×6=54
4th	150	9×6=54
Total	**400**	**108**

14).*When i=14 (assume element from partition is 1032)*

Number of elements in 14th Partition are 3

Assumed row for 1032	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	44	3.3333×3=10
2nd	44	3.3333×3=10
3rd	44	1.3333×3=4
4th	44	1.3333×3=4
Total	**176**	**28**

15).*When i=15 (assume element from partition is 0003)*

Number of elements in 15th Partition are 12

Assumed row for 0003	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	1189	40.3333×12=484
2nd	205	0×12=0
3rd	205	0×12=0
4th	697	19.3333×12=232
Total	**2296**	**716**

16).*When i=16 (assume element from partition is 0022)*

Number of elements in 16th Partition are 12

Assumed row for 0022	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	377	15.5×12=186
2nd	87	0×12=0
3rd	319	8×12=96
4th	87	0×12=0
Total	**870**	**282**

17).*When i=17 (assume element from partition is 0023)*

Number of elements in 17th Partition are 12

Assumed row for 0023	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	170	17.8333×12=214
2nd	51	0×12=0
3rd	170	7.5×12=90
4th	187	5.5×12=66
Total	**578**	**370**

18).*When i=18 (assume element from partition is 0000)*

Number of elements in 18th Partition are 4

Assumed row for 0000	**Total number of composition to be verified**	**contribution towards number of associative**
Ist	80	25.75×4=103
2nd	20	0×4=0
3rd	20	0×4=0
4th	20	0×4=0
Total	**140**	**103**

19).*When i=19 (assume element from partition is 0123)*

Number of elements in 19th Partition are 1

Assumed row for 0123	Total number of composition to be verified	contribution towards number of associative
Ist	1	1×1=1
2nd	1	0×1=0
3rd	1	0×1=0
4th	1	0×1=0
Total	**4**	**1**

Summary of Section

Partition -No	Total contribution towards number of associative binary operations of partition	number of composition tables verified
1	12	4
2	48	253
3	72	229
4	48	607
5	12	535
6	744	3231
7	264	586
8	72	611
9	252	687
10	72	346
11	180	592
12	108	496
13	108	400
14	28	176
15	716	2296
16	282	870
17	370	578
18	103	140
19	1	4
Total	**3492**	**12641**

VIII. CONCLUSION

The conclusion of this paper is that among the 4,294,957,296 different operations on a Four-element set, S={0,1,2,3}, there are exactly 3492 operations which are associative and only 12641 composition tables are to be verified out of 4,294,957,296. In other words, there exist exactly 3492 three-element semi groups.

IX. FUTURE WORK

One can find out Associative binary Operations on a n-Element Set by using same Algorithm , which we have used for Four-element and also verify the one of the result for five element set 183732 in [3].

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